

ON APPROXIMATELY COHEN-MACAULAY BINOMIAL EDGE IDEAL

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ABSTRACT. Binomial edge ideals I_G of a graph G were introduced by [4]. They found some classes of graphs G with the property that I_G is a Cohen-Macaulay ideal. This might happen only for few classes of graphs. A certain generalization of being Cohen-Macaulay, named approximately Cohen-Macaulay, was introduced by S. Goto in [3]. We study classes of graphs whose binomial edge ideal are approximately Cohen-Macaulay. Moreover we use some homological methods in order to compute their Hilbert series.

INTRODUCTION

Let K denote a field. Let G denote a connected undirected graph over the vertices labeled by $[n] = \{1, 2, \dots, n\}$. Let $R = K[x_1, \dots, x_n]$ denote the polynomial ring in n variables over the field K . The edge ideal I of G is generated by all $x_i x_j$, $i < j$, such that $\{i, j\}$ forms an edge of G . This notion was studied by Villarreal [7] where it is also discussed under which circumstances R/I is a Cohen-Macaulay ring. It seems to be hopeless to characterize all of the graphs G such that R/I is a Cohen-Macaulay ring.

In a similar way one might define the binomial edge ideal $I_G \subseteq S = K[x_1, \dots, x_n, y_1, \dots, y_n]$. It is generated by all binomials $x_i y_j - x_j y_i$, $i < j$, such that $\{i, j\}$ is an edge of G . In the paper of V. Ene, J. Herzog, T. Hibi [2], the authors start with the systematic investigation of I_G . There is a primary decomposition of I_G . Moreover there are examples such that S/I_G is a Cohen-Macaulay ring. Furthermore, the authors believe that it is hopeless to characterize those G such that the binomial edge ideal is Cohen-Macaulay in general.

A generalization of the notion of a Cohen-Macaulay ring was introduced by S. Goto [3] under the name approximately Cohen-Macaulay (see also the Definition 1.4). Not so many examples of approximately Cohen-Macaulay rings are known. Note that they are not a domain. In the present paper we collect a few graphs G such that the associated ring S/I_G is an approximately Cohen-Macaulay ring, where I_G denotes the binomial edge ideal.

In fact, we give a complete characterization of all connected trees whose binomial edge ideal are approximately Cohen-Macaulay. Such trees are described as 3-star like. We also prove that the cycle of length n is approximately Cohen-Macaulay.

It is well known fact that the canonical module $\omega(S/I)$ of a Cohen-Macaulay ring is a Cohen-Macaulay module. The converse is not true. Examples of non Cohen-Macaulay rings such that $\omega(S/I)$ is Cohen-Macaulay are approximately Cohen-Macaulay rings (as follows by the definition). So we investigate the canonical modules and the modules

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of deficiency. As applications of our investigations we study the Hilbert series of our examples.

The paper is organized as follows: In the first section there is a summary of preliminary results. In section 2 we give the characterization of approximately Cohen-Macaulay binomial edge ideals for trees. In section 3 we prove that the binomial edge ideal of any cycle is approximately Cohen-Macaulay. In addition we compute the Hilbert series of the corresponding ideals. In order to do that for the n -cycle, we include some investigations on the canonical module of the binomial edge ideal of a complete graph.

1. PRELIMINARY RESULTS

In this section we will fix the notations we use in the sequel. Moreover we summarize auxiliary results that we need in our paper. We have tools from combinatorial algebra, canonical modules and commutative algebra. In general we fix an arbitrary field K and the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ in $2n$ variables x_1, \dots, x_n and y_1, \dots, y_n .

(a) Combinatorial Algebra.

By G a graph on the vertex set $[n]$ we always understand a connected simple graph. The binomial edge ideal $J_G \subset S$ is defined as the ideal generated by all binomials $f_{ij} = x_i y_j - x_j y_i$, $1 \leq i < j \leq n$, where $\{i, j\}$ is an edge of G . This construction has been found in [4]. For a subset $T \subset [n]$ let be $G_{[n] \setminus T}$ the graph obtained from G by deleting all vertices that belongs to T . Let $c = c(T)$ denote the number of connected components of $G_{[n] \setminus T}$. Let G_1, \dots, G_c denote these components of $G_{[n] \setminus T}$. Let \tilde{G} denotes the complete graph on the vertex set of G . Define

$$P_T(G) = (\cup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(T)}})$$

where \tilde{G}_i , $i = 1, \dots, c = c(T)$, denotes the complete graph on the vertex set of the connected component of G_i . Then $P_T(G) \subset S$ is a prime ideal of height $n + |T| - c$, where $|T|$ denotes the number of elements of T . Moreover

$$J_G = \cap_{T \subseteq [n]} P_T(G).$$

For these and related facts we refer to [4]. For us the following lemma is important.

Lemma 1.1. *With the previous notations it follows that $J_G \subset P_T(G)$ is a minimal prime if and only if either $T = \emptyset$ or $T \neq \emptyset$ and for each $i \in T$ we have that $c(T \setminus \{i\}) < c(T)$.*

For the proof see corollary 3.9 of [4]. It is noteworthy to say that J_G is the intersection of prime ideals. That is, S/J_G is reduced. Finally we observe that S/J_G is a graded ring with natural grading induced by the grading of S .

(b) Canonical Modules.

Let S as above denote the polynomial ring. Let M denote the finitely generated graded S -module. For some of our arguments we use a few basic facts about duality. In the following we introduced the modules of deficiency related to M .

Definition 1.2. For an integer i define

$$\omega^i(M) := \text{Ext}_S^{2n-i}(M, S)$$

the i -th module deficiency to M . Note that $\omega^i(M) = 0$ for $i < \text{depth}(M)$ and $i > \dim(M) = d$. For $i = d$ we call

$$\omega(M) = \omega^d(M)$$

the canonical module of M .

These modules have been introduced in [6]. See also [6] for some properties of $\omega^i(M)$ and related facts. In particular we have

$$\dim(\omega(M)) = \dim(M)$$

and $\dim(\omega^i(M)) \leq i$ for all $0 \leq i < d$.

Lemma 1.3. With the previous notation let $I \subset S$ be the homogenous ideal then :

(1) There is a natural homomorphism

$$S/I \rightarrow \omega(\omega(S/I)) \cong \text{Hom}(\omega(S/I), \omega(S/I))$$

which is an isomorphism if S/I is a Cohen-Macaulay ring.

(2) Suppose that S/I is Cohen-Macaulay ring with $\dim(S/I) > 0$. Suppose that $\omega(S/I)$ is an ideal of S/I then $(S/I)/\omega(S/I)$ is a Gorenstein ring with $\dim(S/I) - 1$.

For the proof we refer to [6] and [1].

(c) Approximately Cohen-Macaulay rings.

For our purpose here we need a certain generalization of Cohen-Macaulay rings that was originally introduced by S.Goto (see [3]) in the case of local rings.

Definition 1.4. Let R denote a commutative ring of finite dimension d and $I \subset R$ an ideal. Then

(1) $\text{Assh}_R(R/I) = \{p \in \text{Ass}_R(R/I) : \dim R/p = \dim R/I\}$.

(2) $U_R(I) = \bigcap_{p \in \text{Assh}_R(R/I)} I(p)$ where $I = \bigcap_{p \in \text{Ass}_R(R/I)} I(p)$ denotes a minimal primary decomposition of the ideal I .

That is $U_R(I)$ describes the equidimensional part of the primary decomposition of the ideal I .

As an analogue to Goto's notion of approximately Cohen-Macaulay rings we define a graded version of it.

Definition 1.5. Let $R = \bigoplus_{i \geq 0} R_i$ denote a standard K -algebra, where $K = R_0$. It is called approximately Cohen-Macaulay if $R_+ = \bigoplus_{i > 0} R_i$ contains a homogeneous element x such that $R/x^n R$ is a Cohen-Macaulay ring of $\dim(R) - 1$ for all $n \geq 1$.

A characterization of approximately Cohen-Macaulay rings can be done by the following theorem.

Theorem 1.6. Let $I \subset S$ denote a homogenous ideal and $d = \dim(S/I)$. Then the following conditions are equivalent.

- (i): S/I is approximately Cohen-Macaulay.
- (ii): $S/U_S(I)$ is a d -dimensional Cohen-Macaulay ring and $\text{depth}(S/I) \geq d - 1$.
- (iii): $\omega^d(S/I)$ is Cohen-Macaulay module of dimension d and $\omega^{d-1}(S/I)$ is either zero or a $(d - 1)$ -dimensional Cohen-Macaulay module.

Proof. In the case of local ring the above theorem was proved by Goto in the paper of [3]. The proof in the graded case follows by the same arguments. We have to adopt the graded situation from the local one. We omit the details. \square

Remark 1.7. A necessary condition for S/I to be an approximately Cohen-Macaulay ring is that $\dim S/p \geq \dim S/I - 1$ for all $p \in \text{Ass}(S/I)$

2. TREES

In this section we will characterize all trees which the property that the associated binomial edge ideal defines an approximately Cohen-Macaulay ring. A graph G is called tree if it is connected and has no cycles. The simplest tree is the line. The binomial edge ideal of a line is a complete intersection. So it defines a Cohen-Macaulay ring (see [4]). In our consideration we do not consider the line in detail. To classify all the trees that are approximately Cohen-Macaulay, we have to introduce a new terminology called 3-star like trees.

Definition 2.1. A tree G is called 3-star like if there are no vertices of degree ≥ 4 and if either there is at most one vertex of degree 3 or there is at most one edge with the property that both of its vertices are of degree 3. Then the line is of course 3-star like. Other types of 3-star like trees (see the figures 1 and 2).

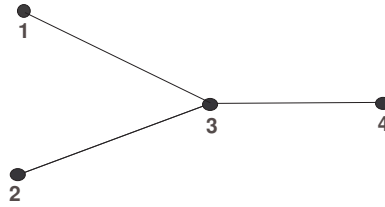


FIGURE 1.

Example 2.2. Consider the simplest example of the 1st type of 3-star like tree as shown in figure 1. Its binomial edge ideal is approximately Cohen-Macaulay of dimension 6, depth 5 and the Hilbert Series is

$$H(S/J_G, t) = \frac{1 + 2t - 2t^3}{(1 - t)^6}.$$

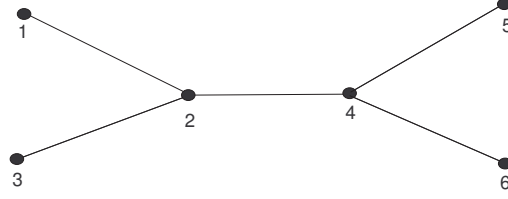


FIGURE 2.

Example 2.3. The simplest example in case of 2nd type of 3-star like tree is shown above. Its binomial edge ideal is approximately Cohen-Macaulay of dimension 8, depth 7 and the Hilbert Series is

$$H(S/J_G, t) = \frac{1 + 4t + 5t^2 - 3t^4}{(1 - t)^8}.$$

In order to prove the approximately Cohen-Macaulay property of binomial edge ideals we need the following construction principle. It will be useful also in different circumstances.

Lemma 2.4. Let G be any connected graph with vertices set $[n]$ having at least one vertex of degree 1, choose one of them and label it by n . Let G' be a graph on vertices set $[n + 1]$ by attaching one edge $\{n, n + 1\}$ to the graph G . Now G is approximately Cohen-Macaulay if and only if G' is approximately Cohen-Macaulay.

Proof. Suppose that J_G and $J_{G'}$ denotes the binomial edge ideal of the corresponding graphs. Let $\dim(S/J_G) = d$ and $\text{depth}(S/J_G) = d - 1$. Now $J_{G'} = (J_G, f)$ where $f = x_n y_{n+1} - x_{n+1} y_n$ and $S' = S[x_{n+1}, y_{n+1}]$, therefore $\dim(S'/J_G) = d + 2$ and $\text{depth}(S'/J_G) = d + 1$.

Now $n \notin T$ for all $T \subseteq [n]$ such that $c(T \setminus \{i\}) < c(T)$. Which implies $x_n, y_n \notin P_T(G)$ for all $P_T(G) \in \text{Ass}(S/J_G)$, hence f is not a zero divisor in S/J_G and is regular. Therefore $\dim(S'/J_{G'}) = d + 1$ and $\text{depth}(S'/J_{G'}) = d$.

Consider the exact sequence

$$0 \rightarrow S'/J_G(-2) \xrightarrow{f} S'/J_G \rightarrow S'/J_{G'} \rightarrow 0.$$

Apply $\text{Hom}(S', \cdot)$ to above sequence we get the long exact sequence

$$\begin{aligned} 0 \rightarrow \omega^{d+2}(S'/J_G) \xrightarrow{f} \omega^{d+2}(S'/J_G)(2) \rightarrow \omega^{d+1}(S'/J_{G'}) \\ \rightarrow \omega^{d+1}(S'/J_G) \xrightarrow{f} \omega^{d+1}(S'/J_G)(2) \rightarrow \omega^d(S'/J_{G'}) \rightarrow 0. \end{aligned}$$

Clearly f is $\omega^{d+2}(S'/J_G)$ -regular. Moreover f is $\omega^{d+1}(S'/J_G)$ -regular since f is S/J_G -regular. So multiplication by f is injective and we get the following two isomorphisms:

$$\omega^{d+1}(S'/J_{G'}) \cong \omega^{d+2}(S'/J_G)/f\omega^{d+2}(S'/J_G)(2)$$

and

$$\omega^d(S'/J_{G'}) \cong \omega^{d+1}(S'/J_G)/f\omega^{d+1}(S'/J_G)(2).$$

Hence $\omega^{d+2}(S'/J_G)$ and $\omega^{d+1}(S'/J_G)$ is of dimension $d+2$ and $d+1$ resp. Therefore both are Cohen-Macaulay if and only if $\omega^{d+1}(S'/J_{G'})$ and $\omega^d(S'/J_{G'})$ are Cohen-Macaulay of dimension $d+1$ and d resp. Therefore from theorem 1.6 $S'/J_{G'}$ is approximately Cohen-Macaulay if and only if S/J_G is approximately Cohen-Macaulay. \square

Corollary 2.5. *Let G be a tree with vertices set $[n]$, then S/J_G is approximately Cohen-Macaulay if and only if G is 3-star like.*

Proof. Suppose the contrary that G is not 3-star like then we have two cases:

Case 1: If G contains at least one vertex of degree $d \geq 4$ say i . Then $T = \{i\}$ and $c(T) = d$ so we have $\dim(S/P_T(G)) = n + d - 1 \geq n + 3$.

Case 2: If G has two vertices of degree 3 which are not adjacent say i and j . Then $T = \{i, j\}$ and $c(T) = 5$. Therefore we have $\dim(S/P_T(G)) = n + 3$.

Therefore in both cases $\dim(S/J_G) \geq n + 3$.

On the other hand $J_{\tilde{G}} \in \text{Ass}(S/J_G)$ and $\dim(S/J_{\tilde{G}}) = n + 1$ so $\text{depth}(S/J_G) \leq n + 1$ and hence S/J_G is not approximately Cohen-Macaulay.

Conversely, in order to prove that any 3-star like tree is approximately Cohen-Macaulay we will use induction on n , the number of vertices. For the case of a line there is nothing to prove. For $n = 4$ and $n = 6$ it is true see example 2.2. and 2.3 above. The general case follows by the construction principle in Lemma 2.4. \square

Now we will discuss some properties of the trees which are approximately Cohen-Macaulay. In Corollary 2.5. we have shown that 3-star like trees with vertices set $[n]$ have dimension $n + 2$ and depth $n + 1$. Now the Hilbert series of 3-star like trees can be easily computed.

Lemma 2.6. *With the notations of Lemma 2.4, We have*

$$H(S'/J_{G'}, t) = (1 - t^2)H(S'/J_G, t).$$

Proof. Consider the exact sequence

$$0 \rightarrow S'/J_G(-2) \xrightarrow{f} S'/J_G \rightarrow S'/J_{G'} \rightarrow 0.$$

Hence we have a required result. \square

Corollary 2.7. *Let G be a 3-star like tree with vertex set $[n]$. Then the Hilbert Series of S/J_G is*

$$\begin{aligned} H(S/J_G, t) &= \frac{(1 + 2t - 2t^3)(1 + t)^{n-4}}{(1 - t)^{n+2}} \quad \text{for } n > 3 \\ \text{and } &\frac{(1 + 4t + 5t^2 - 3t^4)(1 + t)^{n-6}}{(1 - t)^{n+2}} \quad \text{for } n > 5 \text{ respectively} \end{aligned}$$

for the first resp. the second type of 3-starlike trees.

Proof. We will prove this by induction on n . Consider the first case of 3-starlike trees. For $n = 4$ it is true, see example 2.2 resp. 2.3. Suppose the claim is true for n . That is,

$$H(S'/J_G, t) = \frac{(1 + 2t - 2t^3)(1 + t)^{n-4}}{(1 - t)^{n+4}}.$$

Now by Lemma 2.6. we have

$$H(S'/J_{G'}, t) = \frac{(1 + 2t - 2t^3)(1 + t)^{n-3}}{(1 - t)^{n+3}}$$

as required.

Similar arguments might be used in order to calculate the Hilbert Series in the second case of 3-starlike trees. \square

3. CYCLE

In this section we study the algebraic properties of the binomial edge ideal associated to a cycle. A graph G is called cycle if it is a closed directed path, with no repeated vertices other than the starting and ending vertices. We denote the cycle of length n by C and its binomial edge ideal by I_C . To study further properties of I_C or S/I_C we shall need few basic properties from [4]. If

$$I_C = \cap_{T \subseteq [n]} P_T(C)$$

then

$$\dim S/P_T(C) = n + 1 \text{ if } T = \emptyset \text{ and } \dim S/P_T(C) \leq n \text{ if } T \neq \emptyset.$$

Hence

$$\dim(S/I_C) = n + 1 \text{ and } U_S(I_C) = P_\emptyset(C) = J_{\tilde{G}}.$$

Moreover $P_T(C)$ is minimal prime of S/I_C if either (i) $T = \emptyset$ or (ii) if $T \neq \emptyset$ and $|T| > 1$ and no two elements $i, j \in T$ belongs to the same edge of C .

Theorem 3.1. $x_1, y_1 - x_2, \dots, y_{n-1} - x_n, y_n$ is the system of parameters for S/I_C .

Proof. Let $\underline{x} = x_1, y_1 - x_2, \dots, y_{n-1} - x_n, y_n$, $M = S/I_C$ and $I = \text{Ann}(M/\underline{x}M)$. Now $M/\underline{x}M = S/(I_C, \underline{x})$. If we replace $y_1 = x_2, y_2 = x_3, \dots, y_{n-1} = x_n$ in I_C , we get

$$I = (x_1, x_1x_3 - x_2^2, x_2x_4 - x_3^2, \dots, x_{n-2}x_n - x_{n-1}^2, x_n^2, x_nx_2).$$

Clearly $x_1, x_n \in \text{Rad}(I)$, we need to prove that $x_2, \dots, x_{n-1} \in \text{Rad}(I)$. If $x_1, x_1x_3 - x_2^2 \in I$ then $x_2^2 \in I$. It follows that $x_2 \in \text{Rad}(I)$, hence we have a basis of induction. If $x_k \in \text{Rad}(I)$ for $2 \leq k \leq n-2$ then $x_kx_{k+2} - x_{k+1}^2 \in I$ as $k+1 \leq n-1$. Therefore $x_{k+1}^2 \in \text{Rad}(I)$ it follows that $x_{k+1} \in \text{Rad}(I)$. This then implies that $\text{Rad}(I) = (x_1, x_2, \dots, x_n)$. Hence I is m -primary in the ring $K[x_1, \dots, x_n]$ so \underline{x} is the system of parameters of S/I_C . \square

Lemma 3.2. Let I_L be the binomial edge ideal of a line L of length n , $g = x_1y_n - x_ny_1$ and $J_{\tilde{G}}$ be binomial edge ideal of a complete graph on $[n]$ then

- (a): If $I_L = \cap_{T \subseteq [n]} P_T(L)$ then $g \notin P_T(L)$ for $T \neq \emptyset$.
- (b): $I_L = (I_L : g) \cap J_{\tilde{G}}$.
- (c): $I_L : (I_L : g) = J_{\tilde{G}}$.
- (d): $I_L : g = I_L : J_{\tilde{G}}$.

Proof. (a): Because of $I_L = \cap_{T \subseteq [n]} P_T(L)$, it is known from [4] that $P_T(L)$ is minimal prime of I_L if either (i) $T = \emptyset$ or (ii) if $T \neq \emptyset$ and $1, n \notin T$ and if $|T| > 1$ then there are no two elements $i, j \in T$ such that $\{i, j\}$ is an edge of L . If $T = \emptyset$, then $P_T(L) = J_{\tilde{G}}$ is the ideal of complete graph. Now let $T \neq \emptyset$ and $1, n \notin T$. Suppose

that $|T| > 1$ then $g \notin P_T(L)$ because x_1, y_1, x_n, y_n does not belongs to $\cup_{i \in T} \{x_i, y_i\}$ and g does not belongs to any $J_{\tilde{G}}$ for any connected component of $[n] \setminus T$
(b): $I_L : g = \cap_{g \notin P_T(L)} P_T(L)$, Using (a) we have

$$I_L : g = \cap_{T \neq \emptyset} P_T(L)$$

and $I_L = \cap_{T \neq \emptyset} P_T(L) \cap P_{\emptyset}(L)$ therefore,

$$I_L = (I_L : g) \cap J_{\tilde{G}}.$$

(c): Using (b) we have

$$I_L : (I_L : g) = ((I_L : g) \cap J_{\tilde{G}}) : (I_L : g) = J_{\tilde{G}} : (I_L : g).$$

In order to finish we have to prove that $J_{\tilde{G}} = J_{\tilde{G}} : (I_L : g)$. From the definition $J_{\tilde{G}} \subseteq J_{\tilde{G}} : (I_L : g)$. Now for the other inclusion let $I_L : g = (h_1, h_2, \dots, h_r)$, then $J_{\tilde{G}} : (I_L : g) = \cap_{i=1}^r J_{\tilde{G}} : h_i$. Now $J_{\tilde{G}} : h_i = J_{\tilde{G}}$ for at least one i , since $J_{\tilde{G}}$ is a prime ideal and $h_i \notin J_{\tilde{G}}$ for at least one i so $J_{\tilde{G}} : (I_L : g) \subseteq J_{\tilde{G}}$ and we are done.

(d): $I_L : g \subseteq I_L : J_{\tilde{G}}$ is trivial. For another inclusion let $f \in I_L : J_{\tilde{G}}$ then $f J_{\tilde{G}} \subseteq I_L \subseteq P_T(L)$, now $J_{\tilde{G}} \not\subseteq P_T(L)$ for $T \neq \emptyset$, so $f \in P_T(L)$ for $T \neq \emptyset$ which implies $f \in I_L : g$.

□

Definition 3.3. [5] *Two ideals I and J of height g in S are said to be linked if there is a regular sequence α of height g in their intersection such that $I = \alpha : J$ and $J = \alpha : I$.*

It is also known from [5] that I and J are two linked ideals of S then S/I is Cohen-Macaulay if and only if S/J is Cohen-Macaulay.

Lemma 3.4. *Let I_L be the binomial edge ideal of a line of length n then $S/I_L : g$ is Cohen-Macaulay of dimension $n + 1$.*

Proof. I_L is a complete intersection and using (b), (c) and (d) of Lemma 3.2 we have $I_L : g$ and $J_{\tilde{G}}$ are linked ideals. Now it follows from above theorem that $S/I_L : g$ is Cohen-Macaulay because $S/J_{\tilde{G}}$ is Cohen-Macaulay. □

Theorem 3.5. *Any cycle of length $n \geq 3$ is approximately Cohen-Macaulay.*

Proof. First we will compute the depth of S/I_C for $n \geq 3$. From above notations $I_C = (I_L, g)$. Consider the exact sequence

$$0 \rightarrow S/I_L : g(-2) \rightarrow S/I_L \rightarrow S/I_C \rightarrow 0.$$

Now it follows from the Depth's Lemma that

$$\text{depth}(S/I_C) \geq \min\{\text{depth}(S/I_L : g) - 1, \text{depth}(S/I_L)\} = n.$$

Hence $\text{depth}(S/I_C) \geq n$.

Now $S/U_S(I_C) \cong S/J_{\tilde{G}}$, which is $n + 1$ -dimensional Cohen-Macaulay ring, so from Theorem 1.6. S/I_C is approximately Cohen-Macaulay. □

Furthermore we will find the Hilbert series of S/I_C . For this we have to introduce a monomial ideal $M = (x_2 x_3 \cdots x_{n-1}, x_2 x_3 \cdots x_{n-2} y_{n-1}, \dots, x_2 y_3 \cdots y_{n-1}, y_2 y_3 \cdots y_{n-1})$. We need also the expression for the Hilbert series of $S/J_{\tilde{G}}$. Recall the result in [1], if R is

Cohen-Macaulay ring with dimension d and $\underline{x} = x_1, \dots, x_d$ be the homogenous system of parameters of degree 1 then

$$H(R, t) = \frac{H(R/\underline{x}R, t)}{(1-t)^d}.$$

Now in our case $S/J_{\tilde{G}}$ is Cohen-Macaulay ring with dimension $n+1$, $\underline{x} = x_1, y_1 - x_2, \dots, y_{n-1} - x_n, y_n$ is system of parameter of degree 1 of $S/J_{\tilde{G}}$ and

$$S/(\underline{x}S, J_{\tilde{G}}) \cong K[y_1, \dots, y_{n-1}]/(y_1, \dots, y_{n-1})^2$$

therefore $H(S/(\underline{x}S, J_{\tilde{G}}), t) = 1 + (n-1)t$ and we have

$$H(S/J_{\tilde{G}}, t) = \frac{1 + (n-1)t}{(1-t)^{n+1}}.$$

Lemma 3.6. *With the notations above we have*

$$\omega(S/J_{\tilde{G}}) \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

Proof. $J_{\tilde{G}}$ is the ideal of all 2-minors of a generic $2 \times n$ -matrix which implies all 2-minors of a generic $2 \times n$ -matrix are zero in $S/J_{\tilde{G}}$ hence both rows of this matrix are linearly dependent, therefore $S/J_{\tilde{G}} \cong K[x_1, \dots, x_n, x_1t, \dots, x_nt]$. It is known that [1]

$$\omega(S/J_{\tilde{G}}) \cong (x_1, y_1)^{n-2}S/J_{\tilde{G}}$$

therefore

$$\omega(S/J_{\tilde{G}}) \cong (x_1, x_1t)^{n-2}K[x_1, \dots, x_n, x_1t, \dots, x_nt].$$

Next we consider the monomial ideal M in $S/J_{\tilde{G}}$

$$MS/J_{\tilde{G}} \cong (x_2x_3 \cdots x_{n-1}, x_2x_3 \cdots x_{n-1}t, \dots, x_2x_3 \cdots x_{n-1}t^{n-2})K[x_1, \dots, x_n, x_1t, \dots, x_nt].$$

Now after multiplying $\omega(S/J_{\tilde{G}})$ by $x_2x_3 \cdots x_{n-1}$ and $MS/J_{\tilde{G}}$ by x_1 respectively, we see that both are isomorphic that is

$$\omega(S/J_{\tilde{G}}) \cong MS/J_{\tilde{G}} \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

□

Theorem 3.7. *$S/(J_{\tilde{G}}, M)$ is Gorenstein of dimension n .*

Proof. From Lemma 1.3. and 3.6. We have

$$(S/J_{\tilde{G}})/MS/J_{\tilde{G}} \cong S/(J_{\tilde{G}}, M).$$

which is Gorenstein of dimension n .

□

Lemma 3.8. *With the notations above we have*

$$I_L : g = (I_L, M) \text{ and hence } (I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}}).$$

Proof. First we will show that $M \subseteq I_L : g$. Because $I_L \subseteq I_L : g$ it will be enough to prove that

$$(I_L, M)/I_L \subseteq I_L : g/I_L$$

in S/I_L . That is, we have always

$$x_i y_{i+1} \equiv x_{i+1} y_i \pmod{I_L}$$

for $i = 1, \dots, n-1$. Now let $x_2x_3 \cdots x_{n-1} \in M$. With $g = x_1y_n - x_ny_1$, we get

$$x_2x_3 \cdots x_{n-1}g \equiv x_1 \cdots x_{n-1}y_n - x_2 \cdots x_{n-1}x_ny_1 \pmod{I_L}.$$

Now put $x_{n-1}y_n \equiv x_ny_{n-1} \pmod{I_L}$ and $x_{n-2}y_{n-1} \equiv x_{n-1}y_{n-2} \pmod{I_L}$ and so on. After $(n-1)$ steps it follows that

$$x_2x_3 \cdots x_{n-1}g \equiv 0 \pmod{I_L}.$$

This proves that $x_2x_3 \cdots x_{n-1} \in I_L : g$. Similarly $y_2y_3 \cdots y_{n-1} \in I_L : g$. Now take any arbitrary element $x_2 \cdots x_iy_{i+1} \cdots y_{n-1} \in M$ then

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots x_iy_{i+1} \cdots y_n \pmod{I_L}.$$

We have $x_iy_{i+1} \equiv x_{i+1}y_i \pmod{I_L}$. Therefore

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots x_{i-2}(x_{i-1}y_i)(x_{i+1}y_{i+2})y_{i+3} \cdots y_n \pmod{I_L}.$$

Now replace $x_{i-1}y_i = x_iy_{i-1}$ and $x_{i+1}y_{i+2} = x_{i+2}y_{i+1}$ in S/I_L , we have

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots (x_{i-2}y_{i-1})x_iy_{i+1}(x_{i+2}y_{i+3}) \cdots y_n \pmod{I_L}.$$

If we continue such replacements, we get that

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv 0 \pmod{I_L}.$$

So $M \subseteq I_L : g$ hence

$$(M, J_{\tilde{G}}) \subseteq (I_L : g, J_{\tilde{G}}).$$

S/I_L is a Gorenstein ring and $S/I_L \rightarrow S/J_{\tilde{G}}$, therefore from Lemma 1.3

$$\omega(S/J_{\tilde{G}}) \cong \text{Hom}_{S/I_L}(S/J_{\tilde{G}}, S/I_L) \cong (I_L : J_{\tilde{G}})/I_L.$$

Using Lemma 3.2.(d) we get

$$\omega(S/J_{\tilde{G}}) \cong (I_L : g)/I_L.$$

Now from Lemma 3.6 we have that

$$\omega(S/J_{\tilde{G}}) \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

So there are two expressions for $\omega(S/J_{\tilde{G}})$. As the canonical module is unique up to isomorphism. We want to describe an isomorphism. To this end define a map

$$\phi : (J_{\tilde{G}}, M)/J_{\tilde{G}} \rightarrow (I_L : g)/I_L$$

which sends $\sum_{i=1}^{n-1} r_i m_i + J_{\tilde{G}}$ to $\sum_{i=1}^{n-1} r_i m_i + I_L$ where m_1, \dots, m_{n-1} are generators of M . If

$$\sum_{i=1}^{n-1} r_i m_i - \sum_{i=1}^{n-1} r'_i m_i \in J_{\tilde{G}}.$$

Then this implies

$$\sum_{i=1}^{n-1} (r_i - r'_i) m_i \in J_{\tilde{G}} \cap M \subseteq J_{\tilde{G}} \cap (I_L, M) = I_L.$$

This follows because of the inclusion

$$I_L \subseteq J_{\tilde{G}} \cap (I_L, M) \subseteq J_{\tilde{G}} \cap (I_L : g) = I_L$$

as follows by Lemma 3.2 (b). Hence it's a well define map and clearly a homomorphism.

$$\phi \in \text{Hom}_{S/J_{\tilde{G}}}(\omega(S/J_{\tilde{G}}), \omega(S/J_{\tilde{G}})) \cong S/J_{\tilde{G}} \text{ (see Lemma 1.3.)}$$

So any homomorphism $\omega(S/J_{\tilde{G}}) \rightarrow \omega(S/J_{\tilde{G}})$ is given by multiplication by an element of $S/J_{\tilde{G}}$. Because ϕ is a non-zero homomorphism of degree zero it is in fact an isomorphism. That is

$$\text{Im}(\phi) = (I_L, M)/I_L.$$

So finally we get

$$I_L : g = (I_L, M) \text{ and hence } (I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}})$$

□

Lemma 3.9. *The Hilbert series of $S/(I_L : g, J_{\tilde{G}})$ is*

$$H(S/(I_L : g, J_{\tilde{G}}), t) = \frac{1 + (n-1)t - (n-1)t^{n-2} - t^{n-1}}{(1-t)^{n+1}}$$

Proof. It is known from [1] that

$$H(\omega(S/J_{\tilde{G}}), t) = (-1)^{n+1} H(S/J_{\tilde{G}}, \frac{1}{t}).$$

Therefore,

$$H(\omega(S/J_{\tilde{G}}), t) = \frac{(n-1)t^n + t^{n+1}}{(1-t)^{n+1}}.$$

As we know

$$\omega(S/J_{\tilde{G}}) \cong (M, J_{\tilde{G}})/J_{\tilde{G}}.$$

From the above formula the initial degree of $\omega(S/J_{\tilde{G}})$ is n while the initial degree of $(M, J_{\tilde{G}})/J_{\tilde{G}}$ is $n-2$. Therefore by dividing the non-zero divisor t^2 we get the Hilbert series of $(M, J_{\tilde{G}})/J_{\tilde{G}}$

$$H((M, J_{\tilde{G}})/J_{\tilde{G}}, t) = \frac{(n-1)t^{n-2} + t^{n-1}}{(1-t)^{n+1}}.$$

Consider the exact sequence

$$0 \rightarrow (M, J_{\tilde{G}})/J_{\tilde{G}} \rightarrow S/J_{\tilde{G}} \rightarrow S/(M, J_{\tilde{G}}) \rightarrow 0.$$

From Lemma 3.8.

$$(I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}}).$$

Therefore

$$H(S/(I_L : g, J_{\tilde{G}}), t) = H(S/J_{\tilde{G}}, t) - H((M, J_{\tilde{G}})/J_{\tilde{G}}, t).$$

Hence we get the required result. □

Theorem 3.10. *Hilbert series of S/I_C is*

$$H(S/I_C, t) = \frac{(1+t)^{n-1} - t^2(1+t)^{n-1} + (n-1)t^n + t^{n+1}}{(1-t)^{n+1}}.$$

In particular, the multiplicity of S/I_C is $e(S/I_C) = n$.

Proof. Hilbert series of binomial edge ideal of I_L is easy to compute because I_L is a complete intersection generated by $n - 1$ forms of degree 2. Namely we have

$$H(S/I_L, t) = \frac{(1 - t^2)^{n-1}}{(1 - t)^{2n}} = \frac{(1 + t)^{n-1}}{(1 - t)^{n+1}}.$$

Consider the exact sequence

$$0 \rightarrow S/I_L \rightarrow S/I_L : g \oplus S/J_{\tilde{G}} \rightarrow S/(I_L : g, J_{\tilde{G}}) \rightarrow 0.$$

Therefore

$$H(S/I_L : g, t) = H(S/(I_L : g, J_{\tilde{G}}), t) + H(S/I_L, t) - H(S/J_{\tilde{G}}, t).$$

So by using Lemma 3.9. We get

$$H(S/I_L : g, t) = \frac{(1 + t)^{n-1} - (n - 1)t^{n-2} - t^{n-1}}{(1 - t)^{n+1}}.$$

Consider the another exact sequence and replace $(I_L, g) = I_C$

$$0 \rightarrow S/I_L : g(-2) \rightarrow S/I_L \rightarrow S/I_C \rightarrow 0.$$

We have

$$H(S/I_C, t) = H(S/I_L, t) - t^2 H(S/I_L : g, t).$$

and after putting values we get the desired formula. \square

REFERENCES

- [1] W. Burns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, 1993.
- [2] V. Ene, J. Herzog, T. Hibi, Cohen Macaulay Binomial edge ideals, Nagoya Math. J. 204 (2011), 57-68.
- [3] S Goto, Approximately Cohen-Macaulay Rings. J. Algebra 76 (1982), 214-225.
- [4] J. Herzog, T. Hibi, F. Hreinsdotir, T.Kahle, J, Rauh, Binomial edge ideals and conditional independence statements, Adv. Appl. Math. 45 (2010), 317-333.
- [5] C. Peskine and L. Szpiro, Liaison des variétés algébriques. I, Inv. math, 26 (1974), 271-302.
- [6] P. Schenzel, On the use of local cohomology in algebra and geometry. Elias, J. (ed.) et al., Six lectures on commutative algebra. Lectures presented at the summer school, Bellaterra, Spain, July 16-26, 1996. Basel: Birkhuser. Prog. Math. 166 (1998), 241-292.
- [7] R. H. Villarreal, Monomial Algebras, New York: Marcel Dekker Inc. (2001).

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